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BUBBLY FLOW—II

MODELLING VOID FRACTION WAVES

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Abstract—The propagation of voidage waves in vertical bubbly flow is studied using the realistic model of Pauchon and the corresponding simplified model of Lisseter & Fowler. The kinematic wave speed associated with the simple model is found to give quite good predictions of voidage wave speeds. The kinematic wave speeds are also seen to compare well with the characteristic speeds of the full realistic model. Finally, the transition from bubbly to slug flow is discussed in terms of the characteristic and kinematic speeds. It is shown that the liquid slugs in slug flow can be modelled as regions of bubbly flow moving into the preceding regions of falling film flow at the appropriate shock speed.

Key Words: bubbly flow, two-phase flow, mathematical model, void fraction waves, void fraction, slug flow, kinematic wave speed, characteristic speed

1. INTRODUCTION

Increasingly, there is a demand for realistic two-fluid equations to model the transient behaviour of multiphase flow in cooling systems, oil wells and boilers. A time-dependent model is needed to see if a cooling system for a nuclear reactor is susceptible to density wave oscillations. The occurrence of such oscillations would affect the reactor's safe operation. By predicting the flow conditions under which oscillations might occur, such scenarios may be avoided.

For multiphase flow in pipes, a transient can be characterized as a perturbation in the void fraction travelling through the pipes. In this paper we shall focus on the speed of void fraction perturbations in vertical bubbly flow. We shall also look briefly at the formation and propagation of voidage shocks in slug flow.

Cooling systems for nuclear reactors are made up of a large number of pipes. There is therefore an advantage in deriving two-fluid equations that are simple in form. Such equations will aid the efficiency of computer calculations. Simple equations will also enable an analytical investigation of the system, as presented in this paper. The result is a greater physical understanding of the system's behaviour.

In Part I of this paper (Lisseter & Fowler 1992, this issue, pp. 195–204) starting from the set of two-fluid equations for vertical bubbly flow formulated by Pauchon (1987), we have arrived at a suitably simple set. This set was simplified in a rigorous manner by non-dimensionalizing the equations and calculating the sizes of the various terms. We defined dimensionless quantities (as indicated by the superscript "+") in the following manner:

$$z^+ = \frac{z}{L}, \qquad p^+ = \frac{(p - p_{\rm in})}{(\rho_{\rm L} g L)}, \qquad \epsilon_{\rm L}^+ = \epsilon_{\rm L}, \qquad \epsilon_{\rm G}^+ = \frac{\epsilon_{\rm G}(\rho_{\rm G} G_{\rm Lin})}{(\rho_{\rm L} G_{\rm Gin})},$$

$$u_{\rm L}^+ = \frac{u_{\rm L}\rho_{\rm L}}{G_{\rm Lin}}, \qquad u_{\rm G}^+ = \frac{u_{\rm G}\rho_{\rm L}}{G_{\rm Lin}}, \qquad t^+ = \frac{t G_{\rm Lin}}{(\rho_{\rm L} L)}, \qquad [1]$$

where $p_{\rm in}$ is the pressure at the flow inlet, $G_{K\rm in}$ are the inlet mass fluxes and L is the length of the tube. These definitions ensured that the dimensionless variables were of $O(1)$ under the

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experimental conditions used by Mercadier (1981) and Micaelli (1982) (which we shall describe later in this paper).

The simple model which Lisseter & Fowler (1992) derived was as follows:

$$\epsilon^+_{G,t^+} + (\epsilon^+_G u^+_G)_{z^+} = 0, \quad [2a]$$

$$-c_1\epsilon^+_{G,t^+} + [(1 - c_1\epsilon^+_G)u^+_L]_{z^+} = 0 \quad [2b]$$

and

$$u^+_G - u^+_L = \left[\frac{(1 - c_1\epsilon^+_G)}{f^+_i}\right]^{1/2} \quad [2c]$$

(see [35a–c] of Part I, where

$$c_1 = \frac{G_{\text{Gin}}}{G_{\text{Lin}}}\frac{\rho_L}{\rho_G}, \quad [3]$$

$$(f^+_i)^{1/2} = [1 + (17.67(1 - c_1\epsilon^+_G)^{9/7})(18.67(1 - c_1\epsilon^+_G)^{3/2})]^{-1}s^{-1}, \quad [4]$$

$$s = \left(\frac{8g}{3\kappa}\right)^{1/2}\frac{\rho_L}{G_{\text{Lin}}} \quad [5]$$

and

$$\kappa = \frac{4}{3}\left[\frac{g(\rho_L - \rho_G)}{\sigma}\right]^{1/2}. \quad [6]$$

Equations [2a] and [2b] are continuity equations for the gas and liquid phases, while [2c] is a momentum equation representing a balance between the buoyancy and drag forces acting on the bubbles. In the next section we shall use [2a–c] to derive a simple wave equation for the dimensionless void fraction, ϵ^+_G.

2. PREDICTING VOIDAGE WAVE SPEEDS USING A KINEMATIC WAVE EQUATION

Mercadier (1981) and Micaelli (1982) have both conducted experiments to measure the speed of voidage waves in upward bubbly flow. Mercadier studied flow through a cylindrical annulus, which was formed of two concentric tubes of outer and inner radii 0.016 and 0.035 m. He used air and water in the test section and operated the system at 20°C and atmospheric pressure. Micaelli also used air and water at 20°C, but performed his experiments at 6 bar. His test section was a vertical tube with a square cross-section (0.02 × 0.02 m). In this section we shall use [2a–c] to predict speeds of voidage waves as measured in their experiments.

In their experiments, Mercadier (1981) and Micaelli (1982) tried to keep the mass fluxes entering the base of their tubes ($G_{K\text{in}}$) steady. The appropriate boundary conditions for our calculations are thus

$$[\epsilon^+_G u^+_G]_{\text{in}} = 1, \quad [(1 - c_1\epsilon^+_G)u^+_L]_{\text{in}} = 1. \quad [7]$$

If the term ϵ^+_{G,t^+} is eliminated between [2a] and [2b] and these two boundary conditions are used to solve the resulting equation, then we get

$$u^+_L = \frac{(1 + c_1 - c_1\epsilon^+_G u^+_G)}{(1 - c_1\epsilon^+_G)}. \quad [8]$$

This equation may be used to eliminate the variable u^+_L from [2c] and so obtain

$$u^+_G = 1 + c_1 + s(1 - c_1\epsilon^+_G)^3 18.67[1 + 17.67(1 - c_1\epsilon^+_G)^{9/7}]^{-1}. \quad [9]$$

Substitution of this expression into the continuity equation for the gas [2a] gives the following non-linear wave equation for the void fraction:

$$\epsilon^+_{G,t^+} + c^+(\epsilon^+_G)\epsilon^+_{G,z^+} = 0, \tag{10}$$

where the dimensionless wave speed, $c^+(\epsilon^+_G)$, is given by

$$c^+ = 1 + c_1 + 18.67s\epsilon_L^{+2}\left[(4\epsilon_L^+ - 3) + \frac{17.67\epsilon_L^{+9/7}(19\epsilon_L^+ - 12)}{7}\right][(1 + 17.67\epsilon_L^+)^{9/7}]^{-2}, \tag{11}$$

where

$$\epsilon_L^+ = (1 - c_1\epsilon_G^+). \tag{12}$$

The dimensional speed $(G_{Lin}/\rho_L)c^+$ is known as a *kinematic* speed. This is because it was based primarily on the continuity equations, with the momentum equations having been reduced to a force balance in a quasi-static approximation. If the higher order terms had been retained in [2c], then a second-order system would have been obtained, and there would be a second characteristic speed at which disturbances propagate. Such a speed would depend on the wavenumber of the disturbance. This can be easily demonstrated if we represent the state of the system by the vector

$$\boldsymbol{\Psi} \triangleq (\epsilon_G, u_L, u_G, p_G)^T,$$

and rewrite [1a–d] of Part I in the form

$$\underline{\underline{\mathbf{A}}}(\boldsymbol{\Psi})\boldsymbol{\Psi}_t + \underline{\underline{\mathbf{B}}}(\boldsymbol{\Psi})\boldsymbol{\Psi}_z = \mathbf{C}(\boldsymbol{\Psi}), \tag{13}$$

where $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$ are second rank tensors and $\mathbf{C}$ is a first rank tensor; all three tensors are functions of the state variable $\boldsymbol{\Psi}$. Under steady, adiabatic flow conditions, the flow $\boldsymbol{\Psi}_0$ satisfies

$$\mathbf{C}(\boldsymbol{\Psi}_0) = \mathbf{0}.$$

To determine the speed of propagation of a small perturbation to the flow:

$$\boldsymbol{\Psi}^* = \tilde{\boldsymbol{\Psi}}e^{ikz + \gamma t}, \tag{14}$$

where

$$\frac{|\boldsymbol{\Psi}^*|}{|\boldsymbol{\Psi}_0|} \ll 1,$$

we linearize [13] about the steady state. We obtain

$$[\gamma\underline{\underline{\mathbf{A}}}(\boldsymbol{\Psi}_0) + ik\underline{\underline{\mathbf{B}}}(\boldsymbol{\Psi}_0) - \underline{\underline{\mathbf{C}}}'(\boldsymbol{\Psi}_0)]\boldsymbol{\Psi} = \mathbf{0}, \tag{15}$$

where $\underline{\underline{\mathbf{C}}}'(\boldsymbol{\Psi}_0)$ is the Jacobian

$$\frac{\partial}{\partial \boldsymbol{\Psi}}\mathbf{C}(\boldsymbol{\Psi}_0).$$

Equation [14] is a dispersion relation for the speed, $(i\gamma/k)$, at which small perturbations would propagate along the tube. The speed is, in general, dependent on the wavenumber k.

Let k_c be the wavenumber for which

$$\frac{i\gamma(k_c)}{k_c} = c. \tag{16}$$

Whitham (1974) has shown that for systems such as that considered here, in which the higher order terms are small, distributions with wavenumbers other than k_c are damped out exponentially with increasing distance along the tube. The result is that the main part of the disturbance is left propagating with velocity c (and is dispersed by the higher order terms in the equations). Therefore, the fate of the four characteristic wave speeds of the original model in [1a–d] of Part I is as follows.

Two are sound speeds, and are infinite for incompressible fluids. Because of the small (singular) effect of the acceleration terms in the liquid momentum equation, the second finite wave speed is rapidly damped, and the effect is that the first-order equation [10] will in practice describe transients. We should therefore expect the kinematic wave speed, c, to compare well with experimental measurements of the speed of voidage waves in vertical bubbly flow.

In the experiments performed by Mercadier (1981) and Micaelli (1982) the voidage waves which travelled along the tube were just small perturbations to the void fraction appropriate to steady, adiabatic flow. The accuracy of [11] may therefore be assessed by using steady-state values for the void fraction, $c_1 \epsilon_G$. A theoretical expression for the void fraction in the steady state has been derived in a previous paper using the same equations as gave [11] (namely [2a–c]). This expression,

$$c_1 \epsilon_G^+ = c_1 \epsilon_{G0} + c_1^2 \epsilon_{G1} + O(c_1^3), \qquad [17]$$

where

$$\epsilon_{G0} = \frac{1}{(1+s)} \qquad [18a]$$

and

$$\epsilon_{G1} = \epsilon_{G0}^2 [2.633(1 - \epsilon_{G0}) - 1], \qquad [18b]$$

could be used consistently with [11] to obtain the kinematic wave speed corresponding to boundary conditions G_{Lin} and G_{Gin}. We are thus able to calculate the kinematic wave speed knowing only the mass fluxes at the flow inlet and information regarding the effective pipe diameter and physical properties of the experimental fluids.

Despite the possibility, outlined above, of using [17] in conjunction with [11] to predict wave speeds, we shall not use that approach here. Instead we shall use values for $c_1 \epsilon_G$ that were measured by Mercadier (1981) and Micaelli (1982) during their experiments. This will enable us later on to compare predictions of [11] directly with values from other expressions, for which no corresponding relation exists. This practice also avoids compounding errors associated with [11] and [17] (although, as shown in Part I, [17] can give very accurate predictions of the void fraction).

Figure 1 shows the theoretical predictions of [11] using experimental values of $c_1 \epsilon_G$ against data collected by Micaelli (1982). A few items of data are omitted from this comparison because they contradicted the assumption made in formulating [2a–c] of the gas rising faster than the liquid. Under the flow conditions corresponding to these neglected points, the bubbles and fluid velocities were distributed in such a manner over the cross-section of the tube that the mean gas velocity measured by Micaelli (1982) was less than the mean liquid velocity. From figure 1 we see that, in

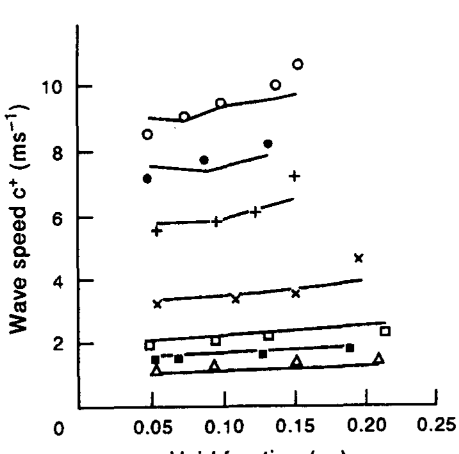

Figure 1. Predictions of the voidage wave speed using [11], compared with the data of Micaelli (1982): G_L(kg m^{-2} s^{-1}) = 8000 (○), 6500 (●), 5000 (+), 3000 (×), 2000 (□), 1500 (■), 1000 (△).

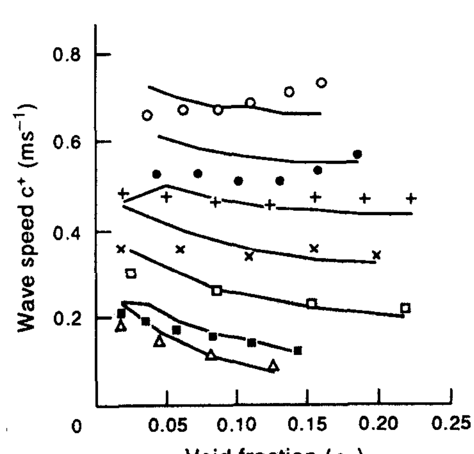

Figure 2. Predictions of the voidage wave speed using [11], compared with the data of Mercadier (1981): G_L(kg m^{-2} s^{-1}) = 912.5 (○), 456.3 (●), 365.0 (+), 273.8 (×), 182.5 (□), 91.25 (■), 0(△).

general, the predictions are quite good, although the theoretical speeds increasingly underestimate the data as the gas flux becomes large. Similar trends are seen when the data of Mercadier (1981) is used (figure 2).

It was noted in the introduction to this paper that Pauchon (1987) used a different and less general expression for the interfacial friction factor. He derived an expression for the kinematic wave speed for the special case when the two-phase Reynolds number, Re_{2p}, is small (<24). Under such conditions

$$c_D \approx \frac{24}{Re_{2p}}.$$

His expression for the kinematic wave speed therefore cannot be as general as [11]. Also, whereas we obtained [11] from a time-dependent system of non-linear equations, Pauchon (1987) linearized his equations for bubbly flow before deriving his kinematic speed. The expression he obtained may be written as a dimensionless wave speed using our notation for comparison with [11]:

$$c^+(\epsilon_G^+) = (1 - c_1\epsilon_G^+)^{-1} + [(\epsilon_G^+)^{-1} + (1 - c_1\epsilon_G^+)^{-1}][1 - 1.5c_1\epsilon_G]. \qquad [19]$$

Pauchon (1987) did not solve his equations to find an expression for the void fraction in the steady state. In figures 3 and 4 we therefore use experimental values for the void fraction in the experiments by Micaelli (1982) and Mercadier (1981) to calculate values from [19] for comparison with experimentally measured wave speeds. The kinematic speed,

$$\left(\frac{G_{Lin}}{\rho_L}\right)c,$$

as calculated from [19] shows a more accurate trend at high gas fluxes than [11], exhibited in figures 1 and 2. However, the predictions of [19] are qualitatively worse that those of our expression at high liquid fluxes.

We conclude that [11] predicts the speed of voidage waves better than the equivalent expression derived by Pauchon. Our expression is also likely to be valid under a wider range of flow conditions than [19].

3. A COMPARISON OF KINEMATIC AND CHARACTERISTIC WAVE SPEEDS

In the previous section we investigated the kinematic wave speed associated with the simplified system [2a–c] and discussed its relation to the speeds at which small perturbations of various

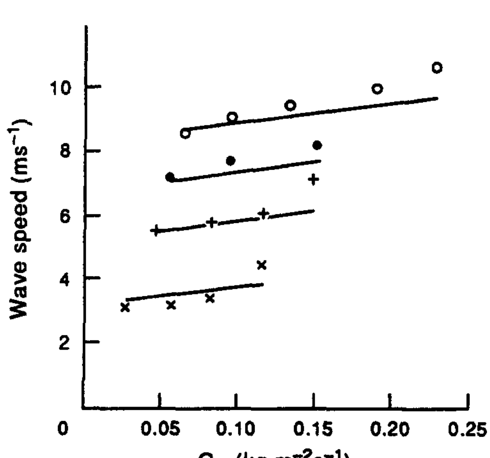

Figure 3. Predictions of the voidage wave speed using [19], compared with the data of Micaelli (1982): symbols as in figure 1.

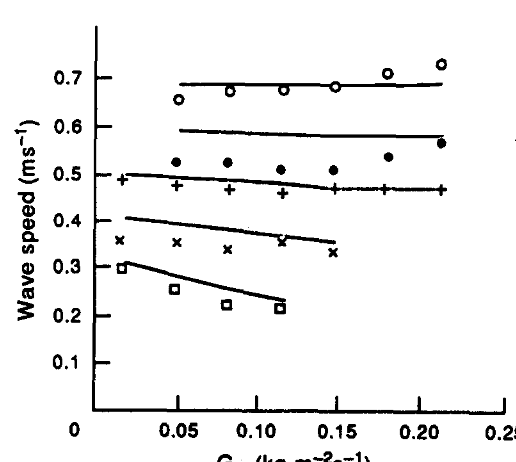

Figure 4. Predictions of the voidage wave speed using [19], compared with the data of Mercadier (1981): symbols as in figure 2.

wavenumbers travelled through the full system, [1a–d] of Part I. In the following paragraphs we shall explore this relationship further, focusing in particular on the behaviour of high wavenumber disturbances.

Recall from section 2 that small perturbations to the steady state travel along the tube at a speed $(i\gamma/k)$, where $\gamma(k)$ satisfies [15]. In the limit, as $k \to \infty$ (high wavenumber), the perturbations will travel at a speed λ which satisfies

$$\det[\underline{\underline{\mathbf{A}}} - \lambda\underline{\underline{\mathbf{B}}}] = 0. \qquad [20]$$

This approximation corresponds to neglecting the algebraic terms in the momentum equations [1c,d] of Part I when calculating the wave speed, but retaining the dynamic terms in those equations. The two solutions to [20], which we shall call λ_+ and λ_-, are known as the characteristic speeds of the system. They represent the fastest and slowest rates at which a small disturbance can propagate stably along the tube (Whitham 1974).

Pauchon & Banerjee (1988) have derived a simple expression for the characteristic speeds of [1a–d] of Part I. This expression for the (dimensional) speeds, $\lambda_\pm$, may be written in the following form:

$$\lambda_\pm = Uu_L^+ + U(u_G^+ - u_L^+) = v_1\left[1 \pm \sqrt{\left(1 + \frac{v_2}{v_1}\right)}\right], \qquad [21]$$

where

$$U = \frac{G_{\mathrm{Lin}}}{\rho_L}, \qquad v_1 = \frac{\epsilon_L^+(C_{\mathrm{VM}} - H)}{(c_1\epsilon_G^+\epsilon_L^+ + C_{\mathrm{VM}})}, \qquad v_2 = \frac{(H - C_{\mathrm{VM}} - \epsilon_L^+ B)}{(C_{\mathrm{VM}} - H)},$$

$$C_{\mathrm{VM}} = \tfrac{1}{2}, \qquad H = \tfrac{1}{4} + \tfrac{1}{5}c_1\epsilon_G^+, \qquad B = \tfrac{1}{4}. \qquad [22]$$

They then went on to derive expressions for the Riemann invariants associated with λ_+ and λ_-. While that associated with λ_- had no readily apparent physical significance, the conserved quantity propagating with speed λ_+ was shown to be closely approximated by the void fraction. This caused Pauchon & Banerjee (1988) to infer that values of λ_+ would be close to the speed at which voidage waves travel through the system. We would therefore suggest that the second characteristic wave speed λ_- is that associated with the small acceleration terms, as discussed in section 2.

In figure 5 we plot values of λ_+ using [21] against wave speed data from the thesis by Micaelli (1982). If this graph is compared with figures 1 and 3, we see that the kinematic wave speeds, as given by [11] and [19] are certainly near to the characteristic wave speeds. Characteristic values for the data of Mercadier (1981) are plotted in figure 6, and a similar comparison can be made with the kinematic wave speed of figures 2 and 4.

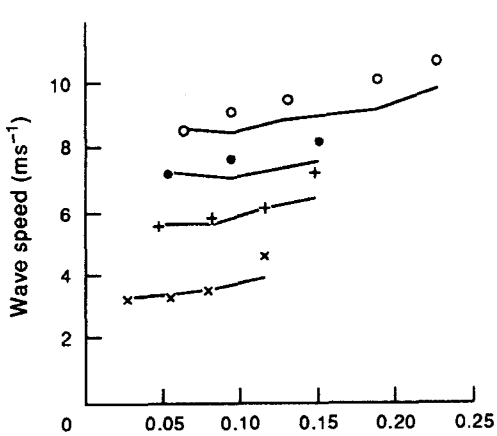

Figure 5. Predictions of the voidage wave speed using characteristic values of [20], compared with the data of Micaelli (1982): symbols as in figure 1.

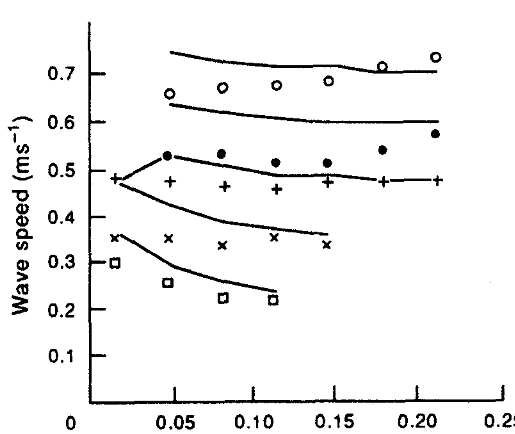

Figure 6. Predictions of the voidage wave speed using characteristic values of [20], compared with the data of Mercadier (1981): symbols as in figure 2.

4. THE TRANSITION TO SLUG FLOW

In this section we will first describe what happens to the characteristic speeds of the full model, [1a–d] of Part I, as the void fraction of bubbly flow is increased by altering the mass fluxes at the flow inlet. We will then investigate how voidage waves might respond to these changing circumstances.

Much work has been carried out on the behaviour of the characteristic speeds as the void fraction in bubbly flow is increased. Pauchon & Banerjee (1988) found that the full model, [1a–d] of Part I, developed complex characteristic speeds at void fractions $\gtrsim 0.42$. This was postulated to indicate a local instability in the void fraction. Such an instability has been observed in experiments by Matuszkiewicz *et al.* (1987) but with void fractions closer to 0.25 than 0.42. They found that a small local disturbance lead to large local values of the void fraction, manifested as bubbles clumping together along the pipe [see also van Beek (1982) and Biesheuvel (1983)]. Once the bubbles had clumped together, surface tension was unable (in an air–water system) to prevent the bubbles from coalescing. In this manner, the large Taylor bubbles which are characteristic of slug flow formed.

After the development of complex characteristic speeds, the equations for bubbly flow break down. They cannot deal with phenomena such as the break up and coalescence of bubbles, and so cannot track the development of the flow through this catastrophe. It is, however, observed that after bubbly flow breaks down, slug flow usually develops. Slug flow is characterized by a rising sequence of large Taylor bubbles interspersed with regions of bubbly flow, known as slugs. In order for the flow in the slugs to be stable, the void fraction in these regions must be less than that which would give complex characteristic speeds. We may therefore apply our equations for bubbly flow to these regions.

Consider how the speed of voidage waves depends on the void fraction. From [11] we obtain

$$\frac{\partial c^+}{\partial \epsilon_L^+} = s\left(\frac{18.67}{49}\right)\epsilon_L^+[1 + 17.67\epsilon_L^{+9/7}]^{-3}$$
$$\times [49(12\epsilon_L^+ - 6) + 17.67\epsilon_L^{+9/7}(654\epsilon_L^+ - 192) + (17.67)^2\epsilon_L^{+18/7}(228\epsilon_L^+ - 60)]. \quad [23]$$

For a given set of flow conditions, the kinematic wave speed therefore attains a maximum value when the local liquid fraction ϵ_L (which, by our choice of scales is identical to ϵ_L^+) is (approximately) equal to 0.29. A similar result would be obtained if we were to use an expression for the wave speed in distorted bubbly flow which was originally derived by Ishii (Hetsroni 1982, pp. 2,109), but which we have cast in our own notation. The dimensionless speed is

$$c^+ = \left(1.2 - 0.2\sqrt{\frac{\rho_G}{\rho_L}}\right)(1 + c_1) + s\left(\frac{11}{4}\epsilon_L^{+7/4} - \frac{7}{4}\epsilon_L^{+3/4}\right), \quad [24]$$

whence

$$\frac{\partial c^+}{\partial \epsilon_L^+} = \frac{7}{16}(\epsilon_L^+)^{-1/4}(11\epsilon_L^+ - 3). \quad [25]$$

With this expression, the wave speed is a maximum when $\epsilon_L^+ \simeq 0.27$.

In slug flow it is observed that the liquid fraction in the slug is significantly larger than 0.29, and the liquid fraction surrounding the Taylor bubbles is either larger than 0.29 or not much less than 0.29. If the Taylor bubbles can be viewed as regions of bubble flow then the speed of voidage waves through the liquid slug will most likely be less than through the Taylor bubbles. Experimental observations of voidage wave speeds accord with this conclusion. During the development of slug flow, Matuszkiewicz *et al.* (1987) detected regions in which the speed of voidage waves was a decreasing function of distance. Whitham (1974, p. 37) notes that under such conditions, the voidage waves would eventually break, and a shock would form. In this case, the shock will form at the front of the slug, where the region of bubbly flow is propagating into an area of higher void fraction. The blunt boundary which is observed to exist between the front of a liquid slug and the rear of the preceding Taylor bubbles seems consistent with this condition.

We can obtain an expression for the speed of this shock from [2a]. The dimensionless shock speed, u_S^+, is a function of the dimensional void fractions ϵ_{GA} and ϵ_{GB} which occur in front of and behind the shock:

$$u_S = \frac{[u_G \epsilon_G]_{\epsilon_{GA}}^{\epsilon_{GB}}}{[\epsilon_G]_{\epsilon_{GA}}^{\epsilon_{GB}}}. \qquad [26]$$

If we assume temporarily that the equations for bubbly flow can be applied to the flow in and around a Taylor bubble, then we may derive a relation for u_S using [9] for u_G:

$$u_S = 1 + c_1 + s(18.67)\frac{\{\epsilon_G(1 - c_1\epsilon_G)^3[1 + 17.67(1 - c_1\epsilon_G)^{9/7}]^{-1}\}_{\epsilon_{GA}}^{\epsilon_{GB}}}{[\epsilon_G]_{\epsilon_{GA}}^{\epsilon_{GB}}}.$$

This expression can be expanded as a power series in the void fraction $c_1\epsilon_G$ (which is, of course, < 1):

$$u_S = 1 + c_1 + s[1 - 1.783c_1(\epsilon_{GA} + \epsilon_{GB}) + 0.656c_1^2(\epsilon_{GA}^2 + \epsilon_{GA}\epsilon_{GB} + \epsilon_{GB}^2) - \cdots].$$

To lowest order in $c_1\epsilon_G$, we have the approximate expression

$$u_S = 1 + c_1 + s. \qquad [27]$$

We can now pause, and comment that a more valid modelling strategy is to treat the flow around the Taylor bubble as a falling annular film. Such an approach has been taken by Seward (1988) in her analysis of slug flow. Then, if flow in the Taylor bubble is indicated by the subscript A (for annular flow), while variables with the subscript B pertain to bubbly flow in the slug, the correct expression for the dimensionless shock speed is

$$u_S = \frac{[u_{GA}\epsilon_{GA} - u_{GB}\epsilon_{GB}]}{[\epsilon_{GA} - \epsilon_{GB}]} = u_{GA} + \frac{[\epsilon_{GA}(u_{GA} - u_{GB})]}{[\epsilon_{GA} - \epsilon_{GB}]}.$$

We see that since experimental observations suggest that $\epsilon_{GA}/[\epsilon_{GA} - \epsilon_{GB}]$ is of order 1, [27] can be regained if $(u_{GA} - u_{GB}) \ll 1$. Fortunately, experimental observations [or even calculations using the model of Seward (1988)] confirm that this inequality does indeed hold, and so we may affirm that the approximate *dimensional* speed of voidage shocks (and hence also of slugs) is

$$U_L u_S = U(1 + c_1) + K, \qquad [28]$$

where $U(1 + c_1)$ is the dimensional volume flux of liquid and gas, and

$$K = \sqrt{2}\left(\frac{\sigma g}{\rho_L - \rho_G}\right)^{1/4},$$

where σ is the surface tension.

In steady slug flow, the shock velocity is equal to the rise velocity of the Taylor bubble modified by the rate at which bubbles are torn from the back of the Taylor bubble and absorbed at the tip of the Taylor bubble. A correlation for the rise velocity of a single Taylor bubble has been obtained in experiments by Nicklin (1962). The expression is (in our notation):

$$U_L u_S = 1.2U_L(1 + c_1) + 0.35\sqrt{(gD)}, \qquad [29]$$

where g is the acceleration due to gravity and D is the diameter of the tube. Collins *et al.* (1978) have also derived an equation of this form by modelling the flow around the nose of the Taylor bubble as if it were inviscid. They considered the first term in [29] to be the liquid velocity on the centreline of the tube, immediately in front of the Taylor bubble, while the second term was taken to represent the terminal velocity of a bubble in stagnant liquid (Wallis 1969; Collins *et al.* 1978). Fernandes *et al.* (1983) and Mao & Dukler (1985) have shown that absorption of bubbles by the Taylor bubbles can be accounted for by modifying the coefficient in [29] from 1.2 to 1.29.

We will now compare predictions from [28] and [29] with data for the speed of voidage waves in slug flow from the thesis by Pauchon (1987). The experiments involved air and water at room

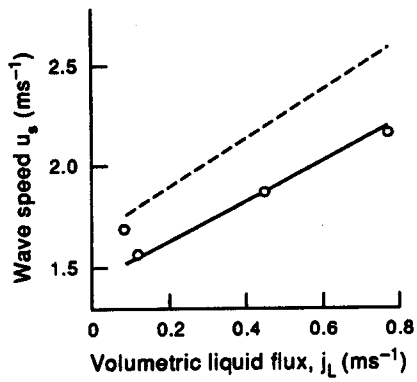

Figure 7. Comparison of the theoretical predictions of voidage wave speed using [28] (——) and [29] (– – –) with the data of Pauchon (1987) (○). The lower curve ([28]) is more accurate.

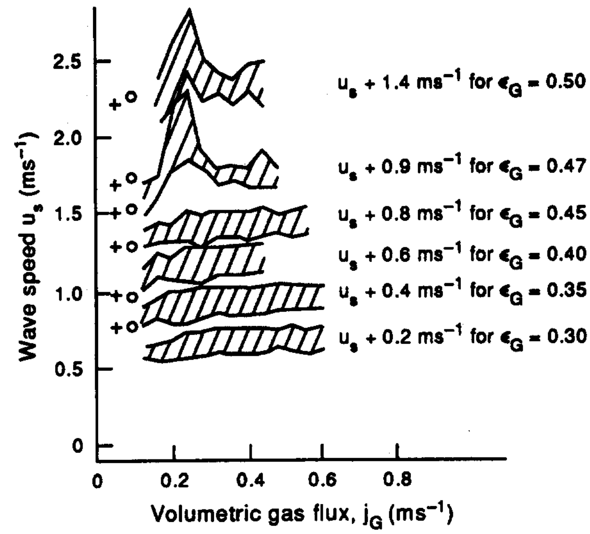

Figure 8. Comparison of the theoretical predictions of voidage wave speed using [28] (+) and [29] (○) with the data of Matuszkiewicz *et al.* (1987); both predictions are comparable, and satisfactory.

temperature flowing through a vertical, cylindrical tube of dia = 0.038 m. From figure 7 we see that both equations give reasonable predictions of the wave speeds, although our equation is slightly better than the experimental correlation of Nicklin (1962). Modifying [29] to account for bubble absorption in fact worsens the predictions.

In figure 8 we compare the predictions of [28] and [29] with data obtained by Matuszkiewicz *et al.* (1987). His apparatus had a square cross-section of side 0.02 m, and nitrogen was bubbled through water at a temperature of 20°C. The density of the gas is calculated to be 1.125 kg m^{-3}, and the surface tension is assumed to be the same as in an air–water flow at room temperature and pressure, namely 0.072 N m^{-2}. In calculating the speed of the Taylor bubble (and thus of the voidage waves) using Nicklin's (1962) correlation, we assume that the appropriate diameter is 0.02 m, rather than the hydraulic diameter of the pipe. Again, both [28] and [29] give reasonable predictions of the speed of voidage waves, although under these conditions [29] appears to give the better results. The peak in the bands for the wave speeds corresponds to values of the voidage wave speed for which the forcing frequency of the void perturbations is close to the characteristic frequency of the slugs.

5. CONCLUSIONS

In this paper we have used a simple but realistic model for upwards bubbly flow to derive a first-order, non-linear differential equation [9] for the propagation of voidage waves. We demonstrated that the kinematic wave speed associated with this equation was able to give quite good predictions of the speed at which voidage waves travelled through the test sections in the experiments by Micaelli (1982) and Mercadier (1981).

Void propagation equations have previously been posed for drift-flux formulations of bubbly flow (Hetsroni 1982, Chap. 2.4). However, those prior equations were linear in nature, whereas ours is non-linear. Also, our expression for the kinematic wave speed was derived in a systematic way from an assumed drag law, whereas the equivalent expression in a drift-flux model would usually have been based on the assumption that the bubbles were affected by the wakes of the upstream bubbles, so that the bubbles moved relative to the mixture velocity, rather than the velocity of the continuous phase (Ishii & Zuber 1979).

We recalled that Pauchon (1987) has also derived a void propagation equation. However, although he derived his equation systematically from an assumed drag law, he did so by linearizing the variables about an assumed steady state. Thus, his wave speed was only applicable to small fluctuations of the void fraction. Furthermore, even if the amplitude of the voidage wave was sufficiently small for a linearized approximation to be valid, Pauchon's expression for the voidage wave speed could not be evaluated without the additional specification of the local void fraction, whereas results from Part I of this paper could enable us to calculate the wave speed merely from a knowledge of the liquid and gas fluxes entering the tube. One final advantage of the model presented here, as compared with that of Pauchon (1987), is that we used a drag law which is applicable over a wide range of flow conditions, including circumstances when the bubbles are distorted, whereas Pauchon chose a friction factor which was only valid at low Reynolds numbers.

In general, our expression [11] for the kinematic wave speed was found to give better predictions of the wave speeds than the expression [19] derived by Pauchon & Banerjee (1988).

In sections 2 and 3 we reviewed the theoretical relation between the kinematic wave speed of [2a–c], the characteristic wave speeds of [1a–d] of Part I and the speed at which voidage waves are observed to travel in bubbly flow.

In the final section of this paper, we followed Pauchon (1987) in interpreting slug flow in terms of the development of complex characteristics speeds, followed by the formation of shocks in the void fraction. We showed that a voidage shock would develop at the rear of a Taylor bubble producing the flat tail characteristic of Taylor bubbles. This result was used to calculate the speed at which voidage waves would travel in slug flow. This speed was found to compare favourably with both the rise velocity of the Taylor bubbles, and also experimental measurements of the speed of voidage waves in slug flow.

We conclude that our model for bubbly flow is able to simulate the propagation of voidage waves. We would therefore have confidence in using it to model transient phenomena such as density wave oscillations.

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